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SMOOTH GREAT CIRCLE FIBRATIONS AND AN APPLICATION TO THE TOPOLOGICAL BLASCHKE CONJECTURE

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Dedicated to Professor Deane Montgomery who taught me transformation groups

ABSTRACT. We study great smooth circle fibrations of round spheres and Blaschke manifolds of the homotopy type of complex projective spaces.

The purpose of the present paper is to prove the following two theorems.

Theorem 2. *Any smooth great circle fibration of the round* $(2n-1)$*-sphere,* $n \neq 3$, *is smoothly equivalent to the Hopf fibration.*

Theorem 3. *Any Blaschke manifold* [1, *p.* 135] *modelled on the complex projective n-space* $\mathbf{C}P^n$, $n \neq 2$, *is diffeomorphic to* $\mathbf{C}P^n$.

We note that the case $n = 3$ in Theorem 2 is valid if the smooth equivalence is replaced by a *topological* equivalence, and then the case $n = 2$ in Theorem 3 is valid if the diffeomorphism is replaced by a *homeomorphism*. (For details, see Gluck-Warner-Yang [3].)

In Sato [8], these theorems have been claimed without smoothness in their conclusions. Although an idea of their proofs is given, it is just impossible to follow. From Sato's response to an inquiry by Frank Warner, we first learned of an outline of his proofs which depend extensively on K-theory. By further inquiries, we suspect that there are gaps and ambiguities. The following is how we see Sato's idea.

Let S^{2n-1} be the unit $(2n-1)$-sphere in the Euclidean $2n$-space $\mathbf{R}^{2n}$ and let $p: S^{2n-1} \to X$ be a smooth great circle fibration of S^{2n-1}. Then there is a natural free smooth action of the circle group G of reals t modulo 1 on S^{2n-1}, say

$$\rho: G \times S^{2n-1} \to S^{2n-1},$$

such that for any $x \in S^{2n-1}$, $\rho(G \times \{x\}) = p^{-1}px$, and for any $(t, x) \in G \times S^{2n-1}$, $x \cdot \rho(t, x) = \cos 2t\pi$. (Strictly speaking, t should read t modulo 1 instead.) Moreover, the action can be smoothly extended to

$$\rho: G \times (\mathbf{R}^{2n} - \{0\}) \to \mathbf{R}^{2n} - \{0\}$$

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such that $\rho(t, x) = |x|\rho(t, x/|x|)$ for any $(t, x) \in G \times (\mathbf{R}^{2n} - \{0\})$.

Let $E_x = T_x\mathbf{R}^{2n}$ be the tangent space of $\mathbf{R}^{2n}$ at x for each $x \in S^{2n-1}$, and let

$$E = \bigcup_{x \in S^{2n-1}} E_x = T\mathbf{R}^{2n}|_{S^{2n-1}},$$

i.e., the tangent bundle of $\mathbf{R}^{2n}$ restricted to S^{2n-1}. Then the extended action ρ induces a free smooth action $d\rho\colon G \times E \to E$, namely the differential of ρ. In order to carry out Sato's idea [8], it is essential to have a complex structure on E such that each E_x may be regarded as the unitary n-space $\mathbf{C}^n$, and for any $(t, v) \in G \times E_x$, $d\rho(t, v) = e^{2t\pi\sqrt{-1}}v \in E_{\rho(t, x)}$. So far, Sato has not made this ambiguity very clear.

In this paper we more or less follow Sato's idea. However, we do not use K-theory at all. Instead, we produce a specific trivialization of E which the proof of Theorem 2 depends on. Theorem 3 is merely a consequence of Theorem 2.

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Throughout this paper, G denotes the circle group of reals modulo 1 and S^{k-1} denotes the unit $(k-1)$-sphere in the Euclidean k-space $\mathbf{R}^k$.

If $\rho_0\colon G \times S^{2n-1} \to S^{2n-1}$ is a free orthogonal action of G on S^{2n-1}, then the orbit space S^{2n-1}/ρ_0 is the complex projective $(n-1)$-space which we denote by $\mathbf{C}P^{n-1}$. Let $\rho\colon G \times S^{2n-1} \to S^{2n-1}$ be a given free smooth action of G on S^{2n-1}. Then the orbit space $X = S^{2n-1}/\rho$ is a closed smooth $(2n-2)$-manifold having the homotopy type of $\mathbf{C}P^{n-1}$, and the projection $p\colon S^{2n-1} \to X$ is a smooth circle fibration of S^{2n-1}. For any $t \in G$,

$$\rho(t)\colon S^{2n-1} \to S^{2n-1},$$

defined by $\rho(t)x = \rho(t, x)$, is a diffeomorphism. Notice that, if $\rho = \rho_0$, then $\rho(1/2)\colon S^{2n-1} \to S^{2n-1}$ is the antipodal map.

Let $\tau\colon E \to S^{2n-1}$ be a trivial smooth $\mathbf{R}^{2n}$ bundle. If $\tilde\rho\colon G \times E \to E$, defined by $\tilde\rho(t)v = \tilde\rho(t, v)$, is a smooth bundle isomorphism satisfying $\tau\tilde\rho(t) = \rho(t)\tau$, then we say that $\tilde\rho$ *covers* ρ. Notice that if $\tilde\rho$ covers ρ, then for any $(t, x) \in G \times S^{2n-1}$, $\tilde\rho(t)$ maps $E_x = \tau^{-1}x$ into $E_{\rho(t)x} = \tau^{-1}\rho(t)x$ and $\tilde\rho(t)\colon E_x \to E_{\rho(t)x}$ is an isomorphism.

Let $\tilde\rho\colon G \times E \to E$ be a free smooth action which covers ρ. If $\varepsilon_1, \ldots, \varepsilon_{2n}\colon S^{2n-1} \to E$ are smooth cross-sections such that $\{\varepsilon_1(x), \ldots, \varepsilon_{2n}(x)\}$ is a basis for E_x for any $x \in S^{2n-1}$, then we call $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ a *trivialization* of E. If $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ is a trivialization of E such that

$$\tilde\rho(1/2)\varepsilon_j(x) = -\varepsilon_j(\rho(1/2)x), \qquad j = 1, \ldots, 2n,$$

for any $x \in S^{2n-1}$, then we say that $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ is a *symmetric trivialization* of E. If $\{\varepsilon_1, \ldots, \varepsilon_{2n}\}$ is a trivialization of E such that

$$\tilde\rho(t)\varepsilon_{2j-1}(x) = (\cos 2t\pi)\varepsilon_{2j-1}(\rho(t)x) + (\sin 2t\pi)\varepsilon_{2j}(\rho(t)x),$$

$$\tilde{\rho}(t)\varepsilon_{2j}(x) = (-\sin 2t\pi)\varepsilon_{2j-1}(\rho(t)x) + (\cos 2t\pi)\varepsilon_{2j}(\rho(t)x), \qquad j = 1, \dots, n,$$

for any $(t, x) \in G \times S^{2n-1}$, then we say that $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is a *good trivialization* of E. Notice that good trivializations are always symmetric. The following result is rather obvious and it indicates the importance of the existence of a good trivialization.

(1) Let $\tilde{\rho}: G \times E \to E$ be a free smooth action covering ρ and let $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ be a good trivialization of E. Then there is a complex structure J on E such that for any $x \in S^{2n-1}$,

$$J\varepsilon_{2j-1}(x) = \varepsilon_{2j}(x), \quad J\varepsilon_{2j}(x) = -\varepsilon_{2j-1}(x), \qquad j = 1, \dots, n.$$

With this complex structure J, E becomes a trivial smooth $\mathbf{C}^n$ bundle $E_{\mathbf{C}}$ such that $\{\varepsilon_{2j-1} | j = 1, \dots, n\}$ is a trivialization of $E_{\mathbf{C}}$ and $\tilde{\rho}: G \times E_{\mathbf{C}} \to E_{\mathbf{C}}$ is given by

$$\tilde{\rho}(t)\varepsilon_{2j-1}(x) = (\cos 2t\pi + J \sin 2t\pi)\varepsilon_{2j-1}(\rho(t)x), \qquad j = 1, \dots, n.$$

The following special case is the only case we find interesting.

(2) Let $\rho: G \times S^{2n-1} \to S^{2n-1}$ be a free smooth action such that $\rho(1/2): S^{2n-1} \to S^{2n-1}$ is the antipodal map, and let $E = T\mathbf{R}^{2n} | S^{2n-1}$ be the tangent bundle of $\mathbf{R}^{2n}$ restricted to S^{2n-1}. Then there is a natural free smooth action $\tilde{\rho}: G \times E \to E$ covering ρ and there is natural symmetric trivialization of E.

Proof. As seen earlier, we have an extended free smooth action

$$\rho: G \times (\mathbf{R}^{2n} - \{0\}) \to \mathbf{R}^{2n} - \{0\}$$

defined by $\rho(t, x) = |x|\rho(t, x/|x|)$. The differential of this extended action, $\tilde{\rho} = d\rho: G \times E \to E$, is a free smooth action of G on E which covers ρ.

Let $\{e_1, \dots, e_{2n}\}$ be the canonical basis of $\mathbf{R}^{2n}$. Then there is a natural trivialization $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ of E such that for any $x \in S^{2n-1}$,

$$\varepsilon_j(x) = e_j \quad \text{at } x, \ j = 1, \dots, 2n.$$

Since $\rho(1/2): S^{2n-1} \to S^{2n-1}$ is an antipodal map, it follows that $\tilde{\rho}(1/2)\varepsilon_j(x) = -\varepsilon_j(\rho(1/2)x)$, $j = 1, \dots, 2n$.

Now we are in a position to prove a technical result which the proof of Theorem 2 depends on.

Theorem 1. *Let $\tilde{\rho}: G \times E \to E$ be a free smooth action covering ρ. If there is a symmetric trivialization of E, then there is a good trivialization of E.*

What we actually need is to apply Theorem 1 to the special case as seen in (2). The justification for the general formulation of Theorem 1 is (3) below which says that in order to prove Theorem 1, we may assume that ρ is a free orthogonal action. The proof of Theorem 1 is long and is divided into many steps.

For $n = 1$, Theorem 1 is obvious. Therefore we assume below that $n > 1$.

(3) It is sufficient to prove Theorem 1 for the special case that ρ is a free orthogonal action ρ_0 of G on S^{2n-1}.

Proof. In order to distinguish ρ and ρ_0, we let S_0^{2n-1} denote the unit $(2n-1)$-sphere under the action ρ_0. Then there are equivariant smooth homotopy equivalences

$$\phi: S_0^{2n-1} \to S^{2n-1}, \qquad \psi: S^{2n-1} \to S_0^{2n-1}$$

such that $\phi\psi: S^{2n-1} \to S^{2n-1}$ is equivariantly homotopic to the identity. Clearly $E' = \phi^* E$ is a trivial smooth $\mathbf{R}^{2n}$ bundle over S_0^{2n-1} and there is a natural free smooth action $\tilde{\rho}: G \times E' \to E'$ which covers ρ_0 and such that ϕ is covered by an equivariant smooth bundle map $\tilde{\phi}: E' \to E$:

$$\begin{array}{ccccc} E & \xrightarrow{\tau} & S^{2n-1} & \xrightarrow{\pi} & S^{2n-1}/\rho = X \\ \uparrow{\scriptstyle\tilde{\phi}} & & \uparrow{\scriptstyle\phi} & & \uparrow \\ E' & \longrightarrow & S_0^{2n-1} & \longrightarrow & S_0^{2n-1}/\rho_0 = \mathbf{C}P^{n-1} \end{array}$$

By hypothesis, there is a symmetric trivialization $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ of E. Then $\{\varepsilon_1\phi, \dots, \varepsilon_{2n}\phi\}$ is symmetric trivializaton of E'. Suppose that Theorem 1 holds for the free orthogonal action ρ_0. Then there is a good trivialization $\{\varepsilon'_1, \dots, \varepsilon'_{2n}\}$ of E'.

Similarly $E'' = \psi^* E'$ is a trivial smooth $\mathbf{R}^{2n}$ bundle over S^{2n-1} and there is a free smooth action $\tilde{\rho}'': G \times E'' \to E''$ which covers ρ and such that ψ is covered by an equivariant smooth bundle map $\tilde{\psi}: E'' \to E'$. Then $\{\varepsilon'_1\psi, \dots, \varepsilon'_{2n}\psi\}$ is a good trivialization of E''.

Since $\phi\psi\phi\psi: S^{2n-1} \to S^{2n-1}$ is equivariantly smoothly homotopic to the identity, it follows from the covering homotopy theorem that E is equivariantly smoothly isomorphic to $(\phi\psi\phi\psi)^* E = (\phi\psi)^* E''$. We know that $\{\varepsilon'_1\psi\phi\psi, \dots, \varepsilon'_{2n}\psi\phi\psi\}$ is a good trivialization of $(\phi\psi)^* E''$. Hence there is a good trivialization of E.

Because of (3), we assume below that ρ is the free orthogonal action ρ_0. We begin with symmetric trivialization of E and then modify it step by step so that we eventually obtain a good trivialization of E.

Let $\mathrm{GL}(2n)$ be the general linear group of nonsingular real $2n \times 2n$ matrices. Then for any trivialization $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ of E, we have a smooth map

$$\phi: G \times S^{2n-1} \to \mathrm{GL}(2n)$$

such that for any $(t, x) \in G \times S^{2n-1}$, the isomorphism $\tilde{\rho}(t): E_x \to E_{\rho(t)x}$ relative to the bases $\{\varepsilon_1(x), \dots, \varepsilon_{2n}(x)\}$ and $\{\varepsilon_1(\rho(t)x), \dots, \varepsilon_{2n}(\rho(t)x)\}$ is represented by $\phi(t, x) \in \mathrm{GL}(2n)$. In terms of matrix multiplication, we have

$$\tilde{\rho}(t)(\varepsilon_1(x), \dots, \varepsilon_{2n}(x)) = (\varepsilon_1(\rho(t)x), \dots, \varepsilon_{2n}(\rho(t)x))\phi(t, x).$$

We call ϕ the *representation* (of the action $\tilde{\rho}$) *associated with the trivialization* $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$.

(4) Let $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ be a trivialization of E and let $\phi\colon G \times S^{2n-1} \to \mathrm{GL}(2n)$ be its associated representation. Then for any $s, t \in G$ and $x \in S^{2n-1}$,

$$\phi(s+t, x) = \phi(s, \rho(t)x)\phi(t, x).$$

Therefore $\phi(0, x) = I$ for any $x \in S^{2n-1}$.

If $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is a symmetric trivialization of E, then for any $(t, x) \in G \times S^{2n-1}$,

$$\phi(1/2 + t, x) = -\phi(t, x)$$

and hence $\phi(1/2, x) = -I$ for any $x \in S^{2n-1}$.

Proof. Since

$$\begin{aligned}(\varepsilon_1(\rho(s+t)x), &\dots, \varepsilon_{2n}(\rho(s+t)x))\phi(s+t, x)\\ &= \tilde{\rho}(s+t)(\varepsilon_1(x), \dots, \varepsilon_{2n}(x))\\ &= \tilde{\rho}(s)\tilde{\rho}(t)(\varepsilon_1(x), \dots, \varepsilon_{2n}(x))\\ &= \tilde{\rho}(s)(\varepsilon_1(\rho(t)x), \dots, \varepsilon_{2n}(\rho(t)x))\phi(t, x)\\ &= (\varepsilon_1(\rho(s)\rho(t)x), \dots, \varepsilon_{2n}(\rho(s)\rho(t)x))\phi(s, \rho(t)x)\phi(t, x),\end{aligned}$$

we infer that $\phi(s+t, x) = \phi(s, \rho(t)x)\phi(t, x)$. Therefore $\phi(s, x) = \phi(s, x) \times \phi(0, x)$ and hence $\phi(0, x) = I$.

If $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is a symmetric trivialization of E, then

$$\begin{aligned}-(\varepsilon_1(\rho(1/2)x), &\dots, \varepsilon_{2n}(\rho(1/2)x))\\ &= \tilde{\rho}(1/2)(\varepsilon_1(x), \dots, \varepsilon_{2n}(x))\\ &= (\varepsilon_1(\rho(1/2)x), \dots, \varepsilon_{2n}(\rho(1/2)x))\phi(1/2, x)\end{aligned}$$

so that $\phi(1/2, x) = -I$. Hence

$$\phi(1/2 + t, x) = \phi(1/2, \rho(t)x)\phi(t, x) = -\phi(t, x).$$

(5) Let $\mathrm{O}(2n)$ be the subgroup of $\mathrm{GL}(2n)$ consisting of orthogonal matrices. Then there is a symmetric trivialization of E whose associated representation is a smooth map $\phi\colon G \times S^{2n-1} \to \mathrm{O}(2n)$.

Proof. There is a Riemannian metric on E which is invariant under the action $\tilde{\rho}$. In fact, such an invariant metric can be obtained by using the action $\tilde{\rho}$ to average any given Riemannian metric on E.

Let $\{\varepsilon'_1, \dots, \varepsilon'_{2n}\}$ be the given symmetric trivialization of E. Then for any $x \in S^{2n-1}$, $\{\varepsilon'_1(x), \dots, \varepsilon'_{2n}(x)\}$ is a basis for E_x so that we may use the Gram-Schmidt process to obtain an orthonormal basis $\{\varepsilon_1(x), \dots, \varepsilon_{2n}(x)\}$ with respect to the invariant Riemannian metric on E. This gives a symmetric trivialization $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ of E.

Now $\tilde{\rho}(t)\colon E_x \to E_{\rho(t)x}$ is an isometry. Hence the representation ϕ associated with $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ maps $G \times S^{2n-1}$ into $\mathrm{O}(2n)$.

The following two results will be needed in our argument.

(6) Let us identify each $A \in \mathrm{O}(k)$ with

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{O}(k+1).$$

Then the inclusion map of $\mathrm{O}(2n)$ into $\mathrm{O}(2n+2)$ induces an isomorphism

$$\pi_k(\mathrm{O}(2n)) \xrightarrow{\cong} \pi_k(\mathrm{O}(2n+2))$$

for $k = 0, \dots, 2n-2$. Let K be the circle subgroup of $\mathrm{O}(2n)$ consisting of

$$A_\theta = \mathrm{diag}\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \dots, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\},$$

and let

$$\mathrm{U}(n) = \{A \in \mathrm{O}(2n) | AA_\theta = A_\theta A \text{ for all } A_\theta \in K\}.$$

Then $\mathrm{U}(n)$ is a subgroup of $\mathrm{O}(2n)$ and the inclusion map of $\mathrm{U}(n)$ into $\mathrm{U}(n+1)$ induces an isomorphism

$$\pi_k(\mathrm{U}(n)) \xrightarrow{\cong} \pi_k(\mathrm{U}(n+1))$$

for $k = 0, \dots, 2n-1$. Let $\mathrm{O}(2n)\mathrm{U}(n)$ be the quotient space of right cosets of $\mathrm{U}(n)$ in $\mathrm{O}(2n)$. Then the inclusion map of $\mathrm{O}(2n)$ into $\mathrm{O}(2n+2)$ induces a map of $\mathrm{O}(2n)/\mathrm{U}(n)$ into $\mathrm{O}(2n+2)/\mathrm{U}(n+1)$ which induces an isomorphism

$$\pi_k(\mathrm{O}(2n)/\mathrm{U}(n)) \xrightarrow{\cong} \pi_k(\mathrm{O}(2n+2)/\mathrm{U}(n+1))$$

for $k = 0, \dots, 2n-2$.

Proof. The first part follows from the homotopy sequences of the fiber bundles

$$\mathrm{O}(2n) \subset \mathrm{O}(2n+1) \to S^{2n},$$
$$\mathrm{O}(2n+1) \subset \mathrm{O}(2n2) \to S^{2n+1}.$$

The second part follows from the homotopy sequence of the fiber bundle $\mathrm{U}(n) \subset \mathrm{U}(n+1) \to S^{2n+1}$.

The inclusion map of $\mathrm{O}(2n)$ into $\mathrm{O}(2n+2)$ induces a map of $\mathrm{O}(2n)/\mathrm{U}(n)$ into $\mathrm{O}(2n+2)/\mathrm{U}(n+1)$ which induces a homomorphism of the homotopy sequence of $\mathrm{U}(n) \subset \mathrm{O}(2n) \to \mathrm{O}(2n)/\mathrm{U}(n)$ into that of $\mathrm{U}(n+1) \subset \mathrm{O}(2n+2) \to \mathrm{O}(2n+2)/\mathrm{U}(n+1)$. Hence the third part follows from earlier parts and the five lemma.

(7) Let $\Omega\mathrm{O}(2n)$ be the space of piecewise smooth paths in $\mathrm{O}(2n)$ from the identity to its negative and let $\Omega_0\mathrm{O}(2n)$ be the subspace of $\Omega\mathrm{O}(2n)$ consisting of minimal geodesics. Then there is a natural diffeomorphism of $\mathrm{O}(2n)/\mathrm{U}(n)$ onto $\Omega_0\mathrm{O}(2n)$ and the inclusion map of $\Omega_0\mathrm{O}(2n)$ into $\Omega\mathrm{O}(2n)$ induces an isomorphism

$$\pi_k(\Omega_0\mathrm{O}(2n)) \xrightarrow{\cong} \pi_k(\Omega\mathrm{O}(2n))$$

for $k = 0, \dots, 2n-3$.

Proof. There is a commutative diagram

$$\begin{array}{ccc}
\pi_{k+1}\mathrm{O}(2n) & \longrightarrow & \pi_{k+1}(\mathrm{O}(2n+2)) \\
\downarrow & & \downarrow \\
\pi_k(\Omega\mathrm{O}(2n)) & \longrightarrow & \pi_k(\Omega\mathrm{O}(2n+2)) \\
\uparrow & & \uparrow \\
\pi_k(\Omega_0\mathrm{O}(2n)) & \longrightarrow & \pi_k(\Omega_0\mathrm{O}(2n+2)) \\
\uparrow & & \uparrow \\
\pi_k(\mathrm{O}(2n)/\mathrm{U}(n)) & \longrightarrow & \pi_k(\mathrm{O}(2n+2)/\mathrm{U}(n+1))
\end{array}$$

given as follows. Let $f\colon \mathrm{O}(2n) \to \mathrm{O}(2n+2)$ be the map given in (6). Then f maps the identity I of $\mathrm{O}(2n)$ into the identity $f(I)$ of $\mathrm{O}(2n+2)$, but does not map $-I$ into $-f(I)$. Let

$$\gamma_1(t)\colon [0,\, 1] \to \mathrm{O}(2n+2)$$

be the smooth path given by

$$\gamma_1(t) = \begin{pmatrix} -I & 0 & 0 \\ 0 & \cos t\pi & -\sin t\pi \\ 0 & \sin t\pi & \cos t\pi \end{pmatrix}$$

which is from $f(-I)$ to $-f(I)$. Then we have a map

$$f'\colon \Omega\mathrm{O}(2n) \to \Omega\mathrm{O}(2n+2)$$

defined by $f'(\gamma) = f\gamma \cdot \gamma_1$. Let γ_2 be a preassigned path in $\mathrm{O}(2n)$ from $-I$ to I. Then there is a homotopy equivalence of $\Omega\mathrm{O}(2n)$ into the space of piecewise smooth loops in $\mathrm{O}(2n)$ with base point I which maps $\gamma \in \Omega\mathrm{O}(2n)$ into $\gamma \cdot \gamma_2$. Similarly we have a homotopy equivalence of $\Omega\mathrm{O}(2n+2)$ into the space of piecewise smooth loops in $\mathrm{O}(2n+2)$ with the identity as its base point which maps each $\gamma' \in \Omega\mathrm{O}(2n+2)$ into $\gamma' \cdot \gamma_1^{-1} \cdot f\gamma_2$. This gives the commutativity of the top rectangle. We note that two vertical arrows of the top rectangle are isomorphisms.

The map $f'\colon \Omega\mathrm{O}(2n) \to \Omega\mathrm{O}(2n+2)$ does not map $\Omega_0\mathrm{O}(2n)$ into $\Omega_0\mathrm{O}(2n+2)$. However, f' is homotopic to the map

$$f''\colon \Omega\mathrm{O}(2n) \to \Omega\mathrm{O}(2n+2)$$

such that for any $\gamma\colon [0,\, 1] \to \mathrm{O}(2n)$ in $\Omega\mathrm{O}(2n)$,

$$(f''\gamma)(t) = \begin{pmatrix} \gamma(t) & 0 & 0 \\ 0 & \cos t\pi & -\sin t\pi \\ 0 & \sin t\pi & \cos t\pi \end{pmatrix}.$$

Since f'' maps $\Omega_0\mathrm{O}(2n)$ into $\Omega_0\mathrm{O}(2n+2)$, the middle rectangle is commutative.

It is clear that $K^+ = \{A_{t\pi} | t \in [0, 1]\}$ is a minimal geodesic in $\mathrm{O}(2n)$ from I to $-I$ and that any minimal geodesic in $\mathrm{O}(2n)$ from I to $-I$ is conjugate to K^+. Therefore there is a natural diffeomorphism of $\mathrm{O}(2n)/\mathrm{U}(n)$ onto $\Omega_0\mathrm{O}(2n)$ which maps each $\mathrm{U}(n)A \in \mathrm{O}(2n)/\mathrm{U}(n)$ into $A^{-1}K^+A$. Hence the bottom rectangle is commutative and its two vertical arrows are isomorphisms.

By a theorem of Bott (see [4, p. 136]),

$$\pi_k(\Omega_0\mathrm{O}(2n+2)) \to \pi_k(\Omega\mathrm{O}(2n+2))$$

is an isomorphism for $k \le 2n-2$. Assume now that $k \le 2n-3$. Then in the commutative diagram above, the top horizontal arrow and the bottom horizontal arrow are isomorphisms. Hence our assertion follows.

Let us regard $\mathbf{R}^k$ as a subspace of $\mathbf{R}^{k+1}$ by identifying every $(x_1, \dots, x_k) \in \mathbf{R}^k$ with $(x_1, \dots, x_k, 0) \in \mathbf{R}^{k+1}$. Then

$$S^1 \subset S^3 \subset \cdots \subset S^{2n-3} \subset S^{2n-1},$$
$$\mathbf{C}P^0 \subset \mathbf{C}P^1 \subset \cdots \subset \mathbf{C}P^{n-2} \subset \mathbf{C}P^{n-1}.$$

Let $p\colon S^{2n-1} \to \mathbf{C}P^{n-1}$ be the projection and let H^{2i} be the closed upper hemisphere in S^{2i}, $i = 0, \dots, n-1$. Then

$$\partial H^{2i} = S^{2i-1} = p^{-1}\mathbf{C}P^{i-1}$$

and $p\colon H^{2i} - \partial H^{2i} \to \mathbf{C}P^i - \mathbf{C}P^{i-1}$ is a diffeomorphism.

Fix an $i = 0, \dots, n-2$ and let each point $(x_1, \dots, x_{2n})$ of S^{2n-1} be written (x', x'') with $x' = (x_1, \dots, x_{2i+2})$ and $x'' = (x_{2i+3}, \dots, x_{2n})$. Then

$$N_i = \{(x', x'') \in S^{2n-1} | |x'| \ge 1/2\}$$

is an invariant closed tubular neighborhood of S^{2i+1} in S^{2n-1} and

$$q_i\colon N_i \times [0, 1] \to N_i,$$

defined by $q_i((x', x''), t) = (t'x', t''x'')$ with $t' = 1/(1-t+t|x'|)$, $t'^2|x'|^2 + t''^2|x'|^2 = 1$ and $t'' \ge 0$, is an equivariant smooth deformation retraction of N_i into S^{2i+1}.

Let $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ be a symmetric trivialization of E whose existence is guaranteed by the hypothesis of Theorem 1. By (5), we may assume that its associated representation is a smooth map $\phi\colon G \times S^{2n-1} \to \mathrm{O}(2n)$. If x is a point of S^{2n-1} such that $t \mapsto \phi(t, x)$ is an isomorphism of G onto a circle subgroup of $\mathrm{O}(2n)$ conjugate to K, we say that $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is *nice* at x.

(8) Let $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ and $\phi\colon G \times S^{2n-1} \to \mathrm{O}(2n)$ be as above. If $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is nice at $x \in S^{2n-1}$, then $\phi(s, \rho(t)x) = \phi(s, x)$ for any $s, t \in G$,

and hence $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is nice at every point of $p^{-1}px$. Moreover, $K_x = \{\phi(t, x) | t \in G\}$ is a circle subgroup of $O(2n)$ conjugate to K which depends only on px.

Proof. By (4), $\phi(s, \rho(t)x) = \phi(s+t, x)\phi(t, x)^{-1} = \phi(s, x)$. The rest is clear.

(9) There is a symmetric trivialization $\{\varepsilon_1^0, \dots, \varepsilon_{2n}^0\}$ of E which is nice at every point of an invariant closed tubular neighborhood of S^1.

Proof. Let $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ and $\phi: G\times S^{2n-1} \to O(2n)$ be as above. Let $q_0: N_0 \times [0, 1] \to N_0$ be the equivariant smooth deformation retraction of N_0 into S^1 constructed earlier and let X_0 be the closed $(2n-2)$-disk such that $q_0^{-1}H^0 = X_0 \times [0, 1]$. Then there is a map $f: X_0 \to \Omega O(2n)$ such that $f(x)(t) = \phi(t/2, x)$ for any $x \in X_0$. By (7), $\pi_0(\Omega_0 O(2n)) \cong \pi_0(\Omega O(2n))$. Then there is a map

$$F: H^0 \times [0, 1] \to \Omega O(2n)$$

such that $F(H^0, 0) = f(H^0)$ and $F(H^0, 1) \in \Omega_0 O(2n)$. Therefore there is a map

$$F': G \times H^0 \times [0, 1] \to O(2n)$$

defined by

$$F'(t, H^0, s) = \begin{cases} F(H^0, s)(2t) & \text{if } t \in [0, 1/2], \\ -F(H^0, s)(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

Since $F'|G \times H^0 \times \{0, 1\}$ is smooth, we can have an F so that F' is smooth.

Let $\lambda: [0, 1] \to [0, 1]$ be a smooth map such that $\lambda[0, 1/3] = 0$ and $\lambda[2/3, 1] = 1$. Then we have a smooth map $\phi^0: G \times S^{2n-1} \to O(2n)$ defined as follows. For any $(t, x) \in G \times (S^{2n-1} - N_0)$,

$$\phi^0(t, x) = \phi(t, x).$$

For any $t \in G$ and $x = (x', x'') \in N_0$ with $1/2 \le |x'| \le 3/4$

$$\phi^0(t, x) = \phi(t, q_0(x, \lambda(4|x'| - 2))).$$

For any $t \in G$ and $x = (x', x'') \in X_0$ with $3/4 \le |x'| \le 1$,

$$\phi^0(t, x) = F'(t, q_0(x, 1), \lambda(4|x'| - 3)).$$

For any $s, t \in G$ and $x = (x', x'') \in X_0$ with $3/4 \le |x'| \le 1$,

$$\phi^0(s, \rho(t)x) = \phi^0(s+t, x)\phi^0(t, x)^{-1}.$$

It is not hard to see that for any $s, t \in G$ and $x \in S^{2n-1}$,

$$\phi^0(s+t, x) = \phi^0(s, \rho(t)x)\phi^0(t, x).$$

Hence ϕ^0 is the representation associated with the symmetric trivialization $\{\varepsilon_1^0, \dots, \varepsilon_{2n}^0\}$ given as follows. For any $x \in S^{2n-1} - N_0$,

$$(\varepsilon_1^0(x), \dots, \varepsilon_{2n}^0(x)) = (\varepsilon_1(x), \dots, \varepsilon_{2n}(x)).$$

For any $t \in G$ and $x \in X_0$,

$$(\varepsilon_1^0(\rho(t)x), \dots, \varepsilon_{2n}^0(\rho(t)x)) = (\varepsilon_1(\rho(t)x), \dots, \varepsilon_{2n}(\rho(t)x))\phi(t, x)\phi^0(t, x)^{-1}.$$

For any $t \in G$ and $x = (x', x'') \in X_0$ with $11/12 \leq |x'| \leq 1$,

$$\phi^0(t, x) = F'(t, q_0(x, 1), 1).$$

Therefore $\{\varepsilon_1^0, \dots, \varepsilon_{2n}^0\}$ is nice at x and hence is nice at every point of $\rho(G \times \{x\})$.

(10) For any $i = 1, \dots, n-1$, if there is a symmetric trivialization $\{\varepsilon_1^{i-1}, \dots, \varepsilon_{2n}^{i-1}\}$ of E which is nice at every point of an invariant closed tubular neighborhood of S^{2i-1}, then there is a symmetric trivialization $\{\varepsilon_1^i, \dots, \varepsilon_{2n}^i\}$ of E which is nice at every point of an invariant closed tubular neighborhood of S^{2i+1}.

Proof. Let $\phi^{i-1}: G \times S^{2n-1} \to O(2n)$ be the representation associated with $\{\varepsilon_1^{i-1}, \dots, \varepsilon_{2n}^{i-1}\}$. Let $q_i: N_i \times [0, 1] \to N_i$ be the equivariant smooth deformation retraction of N_i into S^{2i+1} constructed earlier and let X_i be the set such that $q_i^{-1}H^{2i} = X_i \times [0, 1]$. Then X_i is diffeomorphic to $H^{2i} \times D^{2n-2-2i}$, where $D^{2n-2-2i}$ is the unit closed $(2n-2-2i)$-disk in $\mathbf{R}^{2n-2-2i}$ and $H^{2i} = H^{2i} \times \{0\}$.

Let $f: X_i \to \Omega O(2n)$ be the map such that $f(x)(t) = \phi^{i-1}(t/2, x)$ for any $x \in X_i$. By hypothesis, $\{\varepsilon_1^{i-1}, \dots, \varepsilon_{2n}^{i-1}\}$ is nice at every point of an invariant closed neighborhood N' of S^{2i-1}. Then $f(\partial H^{2i}) \subset f(H^{2i} \cap N') \subset \Omega_0 O(2n)$ so that $f|_{\partial H^{2i}}$ represents an element ζ of $\pi_{2i-1}(\Omega_0 O(2n))$ whose image in $\pi_{2i-1}(\Omega O(2n))$ is 0. Since $2i-1 \leq 2n-3$, we infer from (7) that $\zeta = 0$. Let N' be so chosen such that $N' \subset N_i$ and $N' \cap H^{2i}$ is diffeomorphic to the product of ∂H^{2i} and a closed interval. Then we have a smooth map $f': H^{2i} \to \Omega_0 O(2n)$ such that $f'|_{(H^{2i} \cap N')} = f|_{(H^{2i} \cap N')}$.

Assume first that $i < n-1$. Then $2i \leq 2n-3$ and f' can be chosen so that $f, f': H^{2i} \to \Omega O(2n)$ are homotopic relative to $H^{2i} \cap N'$. Therefore there is a map

$$F: H^{2i} \times [0, 1] \to \Omega O(2n)$$

such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for any $x \in H^{2i}$, and $F(x, s) = f(x)$ for any $(x, s) \in (H^{2i} \cap N') \times [0, 1]$. Hence we have a map

$$F': G \times H^{2i} \times [0, 1] \to O(2n)$$

defined by

$$F'(t, x, s) = \begin{cases} F(x, s)(2t) & \text{if } t \in [0, 1/2], \\ -F(x, s)(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

Since $F'|G \times H^{2i} \times \{0, 1\}$ is smooth, we can choose an F so that F' is smooth.

Let $\lambda: [0, 1] \to [0, 1]$ be a smooth map such that $\lambda[0, 1/3] = 0$ and $\lambda[2/3, 1] = 1$. Then we have a map $\phi^i: G\times S^{2n-1} \to \mathrm{O}(2n)$ defined as follows. For any $(t, x) \in G \times (S^{2n-1} - N_i)$,

$$\phi_i(t, x) = \phi^{i-1}(t, x).$$

For any $t \in G$ and $x = (x', x'') \in X_i$,

$$\phi^i(t, x) = \begin{cases} \phi^{i-1}(t, q_i(x, \lambda(4|x'| - 2))) & \text{if } 1/2 \le |x'| \le 3/4, \\ F'(t, q_i(x, 1), \lambda(4|x'| - 3)) & \text{if } 3/4 \le |x'| \le 1. \end{cases}$$

For any $s, t \in G$ and $x \in X_i$,

$$\phi^i(s, \rho(t)x) = \phi^i(s+t, x)\phi^i(t, x)^{-1}.$$

Then for any $s, t \in G$ and $x \in S^{2n-1}$,

$$\phi^i(s+t, s) = \phi^i(s, \rho(t)x)\phi^i(t, x).$$

Since $\phi^i|_{G\times(X_i - S^{2i-1})}$ is smooth and since ϕ^i is equal to ϕ^{i-1} on $G \times U$ for some neighborhood U of $S^{2n-1} - \operatorname{int} N_i$, it follows that ϕ^i is smooth on $G \times (S^{2n-1} - S^{2i-1})$. Since $\{\varepsilon_1^{i-1}, \dots, \varepsilon_{2n}^{i-1}\}$ is nice at every point of a neighborhood of S^{2i-1}, it follows from the construction of ϕ^i that it is also smooth on $G \times S^{2i-1}$.

Now we are able to construct a desired symmetric trivialization $\{\varepsilon_1^i, \dots, \varepsilon_{2n}^i\}$ having ϕ^i as its associated representation as we did in the proof of (9).

Assume next that $i = n-1$. Then we have a smooth map $\phi^{n-1}: G\times S^{2n-1} \to \mathrm{O}(2n)$ given as follows. For any $t \in G$ and $x \in H^{2n-2}$,

$$\phi^{n-1}(t, x) = \begin{cases} f'(x)(2t) & \text{if } t \in [0, 1/2], \\ -f'(x)(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

For any $s, t \in G$ and $x \in H^{2n-2}$,

$$\phi^{n-1}(s, \rho(t)x) = \phi^{n-1}(s+t, x)\phi^{n-1}(t, x)^{-1}.$$

Similarly, we have the desired symmetric trivialization $\{\varepsilon_1^{n-1}, \dots, \varepsilon_{2n}^{n-1}\}$ of E having ϕ^{n-1} as its associated representation.

Combining (9) and (10), we obtain

(11) There is a symmetric trivialization of E which is nice at every point of S^{2n-1}.

(12) There is a good trivialization of E.

Proof. By (11), there is a symmetric trivialization $\{\varepsilon_1', \dots, \varepsilon_{2n}'\}$ of E which is nice at every point of S^{2n-1}. Let $\phi': G \times S^{2n-1} \to \mathrm{O}(2n)$ be its associated representation. Then for any $x \in S^{2n-1}$, $t \mapsto \phi'(t, x)$ is an isomorphism of G onto a circle subgroup K_x of $\mathrm{O}(2n)$ conjugate to K.

Let $J\colon S^{2n-1} \to \mathrm{O}(2n)$ be the smooth map defined by $J(x) = \phi'(1/4, x)$. Then $J(\rho(t)x) = J(x)$ for any $(t, x) \in G \times S^{2n-1}$, so that $J(x)^2 = -I$ for any $x \in S^{2n-1}$. Hence J is a complex structure on E. With this complex structure, E becomes a smooth $\mathbf{C}^n$ bundle $E_\mathbf{c}$ such that $\tilde{\rho}\colon G \times E_\mathbf{c} \to E_\mathbf{c}$ is given by

$$\rho(t)(\varepsilon_1'(x), \dots, \varepsilon_{2n}'(x)) = e^{2t\pi\sqrt{-1}}(\varepsilon_1'(\rho(t)x), \dots, \varepsilon_{2n}'(\rho(t)x)).$$

Since $\pi_{2n-2}(\mathrm{U}(n)) = 0$, $E_\mathbf{c}$ is trivial so that it has a trivialization $\{\varepsilon_{2j-1} | j = 1, \dots, n\}$. Then $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ with $\varepsilon_{2j} = J\varepsilon_{2j-1}$, $j = 1, \dots, n$, is a trivialization of E such that

$$\rho(t)\varepsilon_{2j-1}(x) = (\cos 2t\pi)\varepsilon_{2j-1}(\rho(t)x) + (\sin 2t\pi)\varepsilon_{2j}(\rho(t)x),$$

$$\rho(t)\varepsilon_{2j}(x) = (-\sin 2t\pi)\varepsilon_{2j-1}(\rho(t)x) + (\cos 2t\pi)\varepsilon_{2j}(\rho(t)x), \qquad j = 1, \dots, n.$$

Hence $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ is a good trivialization of E. This proves (12) as well as Theorem 1.

Now we prepare ourselves to prove Theorem 2.

Let $p\colon S^{2n-1} \to X$ be a free smooth great circle fibration of S^{2n-1}. Then there is a natural free smooth action $\rho\colon G \times S^{2n-1} \to S^{2n-1}$ such that $\rho(G \times \{x\}) = p^{-1}px$ for any $x \in S^{2n-1}$, and $x \cdot \rho(t, x) = \cos 2t\pi$ for any $(t, x) \in G \times S^{2n-1}$. Let

$$E = T\mathbf{R}^{2n}|_{S^{2n-1}}.$$

As seen in (2), there is a natural free smooth action $\tilde{\rho} = d\rho\colon G \times E \to E$ covering ρ and there is a natural symmetric trivialization of E. Hence, by Theorem 1, there is a good trivialization $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ of E.

(13) Let $\widehat{E} = E/\tilde{\rho}$ which is a smooth $\mathbf{R}^{2n}$ bundle over X. Then the total Pontrjagin class of $\widehat{E}$ is equal to the total Pontrjagin class of $\mathbf{C}P^{n-1}$.

Proof. Let J' be the complex structure on $\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ such that for any $u \in \mathbf{R}^{2n}$,

$$J'(u, 0) = (0, u), \qquad J'(0, u) = (-u, 0).$$

With the complex structure J', $\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ becomes the unitary $2n$-space $\mathbf{C}^{2n}$ and there is a natural free orthogonal action

$$\rho'\colon G \times (\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}) \to \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$$

such that for any $t \in G$ and $(u, v) \in \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$,

$$\rho'(t)(u, v) = ((\cos 2t\pi)u - (\sin 2t\pi)v, (\sin 2t\pi)u + (\cos 2t\pi)v).$$

Let K be the sphere in $\mathbf{C}^{2n} = \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ of center $(0, 0)$ and radius $\sqrt{2}$. Then there is a natural diffeomorphism of the orbit space $K^* = K/\rho'$ onto $\mathbf{C}P^{2n-1}$.

Let

$$f\colon \mathbf{R}^{2n} - \{0\} \to \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$$

be the smooth imbedding defined by $f(x) = (x, \rho(3/4)x)$. Then f is equivariant with respect to the extended action $\rho: G \times (\mathbf{R}^{2n} - \{0\}) \to \mathbf{R}^{2n} - \{0\}$ and ρ'. In fact, for any $(t, x) \in G \times (\mathbf{R}^{2n} - \{0\})$,

$$\begin{aligned}\rho'(t)f(x) &= \rho'(t)(x, \rho(3/4)x) \\ &= ((\cos 2t\pi)x - (\sin 2t\pi)\rho(3/4)x, (\sin 2t\pi)x + (\cos 2t\pi)\rho(3/4)x) \\ &= (\rho(t)x, \rho(t)\rho(3/4)x) = f(\rho(t)x).\end{aligned}$$

Since $fS^{2n-1} \subset K$, f induces a smooth imbedding $\overline{f}: X \to K^*$. Let

$$T = T(\mathbf{R}^{2n} \oplus \mathbf{R}^{2n})|_{fS^{2n-1}},$$

i.e., the tangent bundle of $\mathbf{C}^{2n} = \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ restricted to fS^{2n-1}. Then the differential of ρ' gives a free smooth action

$$\tilde{\rho}' = d\rho': G \times T \to T$$

and $\widehat{T} = T/\tilde{\rho}'$ is a smooth $\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ bundle over $\overline{f}X = fS^{2n-1}/\rho'$.

Now we construct two smooth bundle imbeddings $h_1, h_2: E \to T$ covering f as follows. Let $\{e_1, \dots, e_{2n}\}$ be the trivialization of E defined by the canonical basis $\{e_1, \dots, e_{2n}\}$ of $\mathbf{R}^{2n}$, that means, for any $x \in S^{2n-1}$,

$$e_j(x) = e_j \quad \text{at } x, \ j = 1, \dots, 2n.$$

Then for any $x \in S^{2n-1}$, the tangent space of $\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ at fx is $T_{fx} = E_x \oplus E_{\rho(3/4)x}$ which has a natural basis

$$\begin{aligned}&\{e_1(x), e_2(x), \dots, e_{2n-1}(x), e_{2n}(x); e_2(\rho(3/4)x), \\ &\quad -e_1(\rho(3/4)x), \dots, e_{2n}(\rho(3/4)x), -e_{2n-1}(\rho(3/4)x)\}.\end{aligned}$$

On the other hand, Theorem 1 provides a good trivialization $\{\varepsilon_1, \dots, \varepsilon_{2n}\}$ of E so that for each $x \in S^{2n-1}$, $\{\varepsilon_1(x), \dots, \varepsilon_{2n}(x)\}$ is a basis for E. We let

$$\begin{aligned}h_1(\varepsilon_{2j-1}(x)) &= (e_{2j-1}(x), e_{2j}(\rho(3/4)x)), \\ h_1(\varepsilon_{2j}(x)) &= (-e_{2j}(x), e_{2j-1}(\rho(3/4)x)), \\ h_2(\varepsilon_{2j-1}(x)) &= (e_{2j-1}(x), -e_{2j}(\rho(3/4)x)), \\ h_2(\varepsilon_{2j}(x)) &= (e_{2j}(x), e_{2j-1}(\rho(3/4)x)), \qquad j = 1, \dots, n.\end{aligned}$$

It is not hard to verify that both h_1 and h_2 are equivariant. For example, for any $t \in G$,

$$\begin{aligned}h_1(\tilde{\rho}(t)\varepsilon_{2j-1}(x)) &= h_1((\cos 2t\pi)\varepsilon_{2j-1}(\rho(t)x) + (\sin 2t\pi)\varepsilon_{2j}(\rho(t)x)) \\ &= (\cos 2t\pi)(e_{2j-1}(\rho(t)x), e_{2j}(\rho(3/4)\rho(t)x)) \\ &\quad + (-\sin 2t\pi)(e_{2j}(\rho(t)x), -e_{2j-1}(\rho(3/4)\rho(t)x)) \\ &= ((\cos 2t\pi)e_{2j-1}(\rho(t)x) + (-\sin 2t\pi)e_{2j}(\rho(t)x), \\ &\quad + (\sin 2t\pi)e_{2j-1}(\rho(t)\rho(3/4)x) + (\cos 2t\pi)e_{2j}(\rho(t)\rho(3/4)x)) \\ &= \tilde{\rho}'(t)(e_{2j-1}(x), e_{2j}(\rho(3/4)x)) \\ &= \tilde{\rho}'(x)h_1(\varepsilon_{2j-1}(x)).\end{aligned}$$

It is easily seen that for any $x \in S^{2n-1}$,

$$T_{fx} = h_1 E_x \oplus h_2 E_x .$$

We infer that $h = (h_1, h_2)\colon E \oplus E \to T$ is an equivariant smooth bundle isomorphism which induces a smooth bundle isomorphism $\overline{h}\colon \widehat{E} \oplus \widehat{E} \to \widehat{T}$.

For the sake of convenience, let us use $\overline{f}$ to identify X with $\overline{f}X \subset K^*$ and use $\overline{h}$ to identify $\widehat{E} \oplus \widehat{E}$ with $\widehat{T}$. Let α be a generator of the second integral cohomology group $H^2(K^*)$ of K^*. It is well known that the total Pontrjagin class of K^* is

$$p_*(K^*) = (1 + \alpha^2)^{2n} \quad \text{(with the understanding that } \alpha^k = 0 \text{ for } k > 2n-1).$$

The inclusion map of X into K^* induces an isomorphism of $H^2(K^*)$ onto $H^2(X)$ so that we may regard α as a generator of $H^2(X)$ as well. Since $\widehat{T}$ is the Whitney sum of the tangent bundle of K^* restricted to X and two trivial $\mathbf{R}^1$ bundles, we infer that

$$p_*(\widehat{T}) = (1 + \alpha^2)^{2n} \quad \text{(with the understanding that } \alpha^k = 0 \text{ for } k > n-1).$$

Therefore

$$p_*(\widehat{E})^2 = p_*(\widehat{E} \oplus \widehat{E}) = p_*(\widehat{T}) = (1 + \alpha^2)^{2n}$$

and hence

$$p_*(\widehat{E}) = (1 + \alpha^2)^{n} \quad \text{(with the understanding that } \alpha^k = 0 \text{ for } k > n-1).$$

Since $\widehat{E}$ is the Whitney sum of the tangent bundle of X and two trivial $\mathbf{R}^1$ bundles, it follows that

$$p_*(X) = p_*(E) = (1 + \alpha^2)^n .$$

Hence the proof is complete.

(14) There is a smooth homotopy equivalence $f\colon X \to \mathbf{C}P^{n-1}$ such that for any $i = 2, \ldots, n-2$, f is transversal to $\mathbf{C}P^i$ and $f^{-1}\mathbf{C}P^i$ is a closed smooth $2i$-manifold of the homotopy type of $\mathbf{C}P^i$.

Proof. Let us begin with an arbitrary smooth homotopy equivalence $f'\colon X \to \mathbf{C}P^{n-1}$. It is well known that f' is smoothly homotopic to a smooth homotopy equivalence $f_2\colon X \to \mathbf{C}P^{n-1}$ which is transversal to $\mathbf{C}P^2$ so that $f^{-1}\mathbf{C}P^2$ is a closed smooth 4-manifold.

As seen in Montgomery-Yang [6], we can alter f_2 by a smooth homotopy which leads to surgeries killing homotopy groups of $f_2\colon f_2^{-1}\mathbf{C}P^2 \to \mathbf{C}P^2$. In fact, we first alter f_2 by a smooth homotopy which leads to surgeries killing the kernel of

$$f_{2*}\colon \pi_0(f_2^{-1}\mathbf{C}P^2) \to \pi_0(\mathbf{C}P^2).$$

Then $f_2^{-1}\mathbf{C}P^2$ is a connected orientable closed smooth 4-manifold. Let K_2 be the nontorsion part of the kernel of $f_{2*}: H_2(f^{-1}\mathbf{C}P^2) \to H_2(\mathbf{C}P^2)$, let $\mathbf{Z}$ be the additive group of integers, and let $f^{-1}\mathbf{C}P^2$ be so oriented that the value of α^2 at the fundamental class $[f_2^{-1}\mathbf{C}P^2]$ of $f_2^{-1}\mathbf{C}P^2$ is 1. Then there is a nonsingular symmetric bilinear function $B: K_2 \times K_2 \to \mathbf{Z}$ such that for any $u, v \in K_2$, $B(u, v)$ is equal to the intersection number of u and v. The signature I_2 of B is an integer which is independent of the choice of f_2.

By (13), $f_2^* p_*(\mathbf{C}P^{n-1}) = p_*(X)$. We infer that

$$f_2^* p_*(\mathbf{C}P^2) = p_*(f_2^{-1}\mathbf{C}P^2).$$

By the signature theorem [5, p. 224], $I_2 = 0$. Hence we can alter f_2 by a smooth homotopy which leads to surgeries killing not only the kernel of $f_{2*}: \pi_0(f_2^{-1}\mathbf{C}P^2) \to \pi_0(\mathbf{C}P^2)$, but also the kernel of $f_{2*}: \pi_i(f_2^{-1}\mathbf{C}P^2) \to \pi_i(\mathbf{C}P^2)$ for $i = 1, 2$. Hence $f_2: f_2^{-1}\mathbf{C}P^2 \to \mathbf{C}P^2$ is a homotopy equivalence.

Now we proceed by induction and assume that for some integer $k = 3, \dots, n-2$, f' is smoothly homotopic to a smooth homotopy equivalence $f_k: X \to \mathbf{C}P^{n-1}$ such that for any $i = 2, \dots, k-1$, f_k is transversal to $\mathbf{C}P^i$, $f_k: f_k^{-1}\mathbf{C}P^i \to \mathbf{C}P^i$ is a homotopy equivalence, and f_k is transversal to $\mathbf{C}P^k$ so that $f^{-1}\mathbf{C}P^k$ is a closed smooth $2k$-manifold. We can always alter f_k by a smooth homotopy such that $f_k|_{f_k^{-1}\mathbf{C}P^{k-1}}$ remains unchanged and the homotopy leads to surgeries killing the kernel of

$$f_{k*}: \pi_i(f_k^{-1}\mathbf{C}P^k) \to \pi_i(\mathbf{C}P^k)$$

for $i < k$. Then

$$f_{k*}: H_i(f_k^{-1}\mathbf{C}P^k) \to H_i(\mathbf{C}P^k)$$

is an isomorphism for $i < k$ and is surjective for $i = k$. Let

$$K_k = \ker H_k(f_k^{-1}\mathbf{C}P^k) \xrightarrow{f_{k*}} H_k(\mathbf{C}P^k)$$

and let $f_k^{-1}\mathbf{C}P^k$ be so oriented that the value of α^k at the fundamental class $[f_k^{-1}\mathbf{C}P^k]$ of $\mathbf{C}P^k$ is equal to 1. Then we have a nonsingular bilinear function $B: K_k \times K_k \to \mathbf{Z}$ such that for any $u, v \in K_k$, $B(u, v)$ is equal to the intersection number of u and v. The bilinear function B is symmetric or antisymmetric according as k is even or odd.

Assume first that k is even. Then the signature I_k of B is an integer which is independent of the choice of f_k. By (13), $f_k^* p_*(\mathbf{C}P^{n-1}) = p_*(X)$. We infer that

$$f_k^* p_*(\mathbf{C}P^k) = p_*(f_k^{-1}\mathbf{C}^k).$$

By the signature theorem, $I_k = 0$. Hence we can alter f_k by a smooth homotopy which leaves $f_k|_{f_k^{-1}\mathbf{C}P^{k-1}}$ unchanged and leads to surgeries killing K_k. This gives a smooth homotopy equivalence $f_k: X \to \mathbf{C}P^{n-1}$ such that for any

$i = 2, \ldots, k$, f_k is transversal to $\mathbf{C}P^i$ and $f_k\colon f_k^{-1}\mathbf{C}P^i \to \mathbf{C}P^i$ is a homotopy equivalence.

Assume next that k is odd. Then B reduced to integers modulo 2 defines an Arf invariant c_k which is an integer modulo 2. Since $\rho(1/2)\colon S^{2n-1} \to S^{2n-1}$ is the antipodal map, S^{2n-1} contains a smooth $(2k+1)$-sphere invariant under $\rho(1/2)$. Therefore $c_k = 0$. Hence we can alter f_k by a smooth homotopy which leaves $f_k|_{f_k^{-1}\mathbf{C}P^{k-1}}$ unchanged and leads to surgeries killings K_k. This again gives a smooth homotopy equivalence $f_k\colon X \to \mathbf{C}P^{n-1}$ such that for any $i = 2, \ldots, k$, f_k is transversal to $\mathbf{C}P^i$, f_k is transversal to $\mathbf{C}P^i$ for any $i = 2, \ldots, k$, and $f_k\colon f_k^{-1}\mathbf{C}P^i \to \mathbf{C}P^i$ is a homotopy equivalence.

The smooth homotopy equivalence $f = f_{n-1}\colon X \to \mathbf{C}P^{n-1}$ is clearly as desired.

(15) For $n \neq 3$, X is diffeomorphic to $\mathbf{C}P^{n-1}$.

Proof. For $n = 1, 2$, the statement is obvious. Therefore we assume that $n > 3$.

By (14), there is a smooth homotopy equivalence $f\colon X \to \mathbf{C}P^{n-1}$ such that for each $i = 2, \ldots, n-2$, f is transversal to $\mathbf{C}P^i$ and $f\colon f^{-1}\mathbf{C}P^1 \to \mathbf{C}P^1$ is a homotopy equivalence. Let

$$X_i = f^{-1}\mathbf{C}P^i, \qquad i = 3, \ldots, n-1.$$

Then X_3 is diffeomorphic to $\mathbf{C}P^3$ (see, for example, Montgomery-Yang [6]).

Suppose that for some $k = 4, \ldots, n-1$, we have already proved that X_{k-1} is diffeomorphic to $\mathbf{C}P^{k-1}$. Then there is a homotopy $2k$-sphere Σ^{2k} such that X_k is diffeomorphic to $\mathbf{C}P^k \# \Sigma^{2k}$.

In $p^{-1}X_k$, $p^{-1}X_{k-1}$ bounds a smooth closed $2k$-disk D^{2k} such that $p\colon D^{2k} - \partial D^{2k} \to X_k - X_{k-1}$ is a diffeomorphism [7]. Moreover,

$$K^{2k} = D^{2k} \cup \rho(1/2)D^{2k}$$

is diffeomorphic to S^{2k} and the closed smooth $2k$-manifold $K^{2k}/\rho(1/2)$ obtained from K^{2k} by identifying every point x with $\rho(1/2)x$ is diffeomorphic to the connected sum of the real projective $2k$-space $\mathbf{R}P^{2k}$ and Σ^{2k}. Since $\rho(1/2)\colon S^{2n-1} \to S^{2n-1}$ is the antipodal map, S^{2k} is invariant under $\rho(1/2)$. Therefore $K^{2k}/\rho(1/2)$ and $S^{2k}/\rho(1/2) = \mathbf{R}P^{2k}$ are h-cobordant (Browder-Livesay [2]). By the S-cobordism theorem, $K^{2k}/\rho(1/2)$ is diffeomorphic to $\mathbf{R}P^{2k}$ so that Σ^{2k} is diffeomorphic to S^{2k}. Hence X_k is diffeomorphic to $\mathbf{C}P^k$.

(16) If M is a Blaschke manifold modelled on $\mathbf{C}P^n$, then there is a smooth great circle fibration $p\colon S^{2n+1} \to M$.

Proof. Let b be a point of M, T_bM the tangent space of M at b, and $\exp\colon T_bM \to M$ the exponential map. If closed geodesics in M are of length

$2l$ and D is the closed disk in T_bM of center 0 and radius l, then $\exp\partial D$ is the cut locus of b which we denote by $\operatorname{cut} b$, $\exp\colon \partial D \to \operatorname{cut} b$ is a smooth great circle fibration, and $\exp\colon D - \partial D \to M - \operatorname{cut} b$ is a diffeomorphism.

Let H^{2n} be the closed upper hemisphere in S^{2n}. Then $\partial H^{2n} = S^{2n-1}$ and there is a diffeomorphism $f\colon H^{2n} \to D$ given as follows. Let us regard T_bM as $\mathbf{R}^{2n}$ and let x_0 be the north pole of S^{2n}. Then every point of H^{2n} can be written $(\cos\theta)x_0 + (\sin\theta)x$ for some $\theta \in [0, \pi/2]$ and $x \in \partial H^{2n}$. We let

$$f((\cos\theta)x_0 + (\sin\theta)x) = (2\theta l/\pi)x.$$

The smooth imbedding $g = (\exp)f\colon H^{2n} \to M$ has the following properties. First, $g\colon S^{2n-1} \to \operatorname{cut} b$ is a smooth great circle fibration. Second, $g\colon H^{2n} - S^{2n-1} \to M - \operatorname{cut} b$ is a diffeomorphism. Hence up to a diffeomorphism, we may regard M as the smooth manifold obtained from H^{2n} by identifying S^{2n-1} with $\operatorname{cut} b$ via the map g.

Let $\rho\colon G \times S^{2n-1} \to S^{2n-1}$ be the natural free smooth action of the circle group G on S^{2n-1} associated with $g\colon S^{2n-1} \to \operatorname{cut} b$ and let points of S^{2n+1} be written (qx, r, s), where $x \in S^{2n-1}$ and q, r, s are real numbers such that $q \geq 0$, $q^2 + r^2 + s^2 = 1$. Then

$$H^{2n} = \{(qx, r, s) \in S^{2n+1} | r \geq 0, s = 0\}.$$

Therefore there is a natural free smooth action $\rho'\colon G \times S^{2n+1} \to S^{2n+1}$ such that for any $t \in G$ and $(qx, r, 0) \in H^{2n}$,

$$\rho'(t, (qx, r, 0)) = (q\rho(t, x), r\cos 2t\pi, r\sin 2t\pi).$$

Moreover, there is a natural identification of the orbit space S^{2n+1}/ρ' with M. Since the projection of S^{2n+1} into M is a smooth great circle fibration, our assertion follows.

Theorem 3 is a consequence of (16) and Theorem 2.

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